Model Checking and Games

Part IV - Model checking LTL

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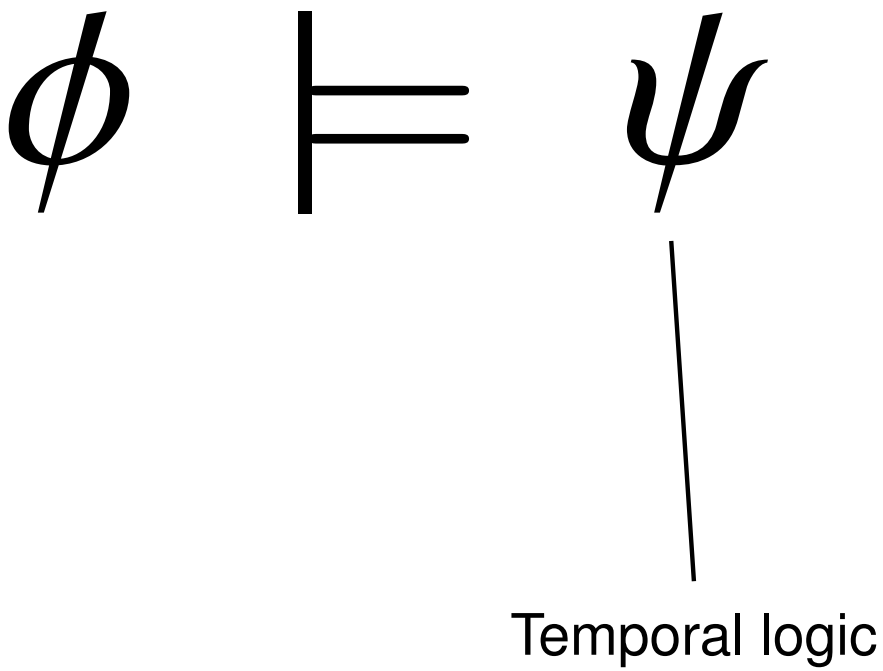
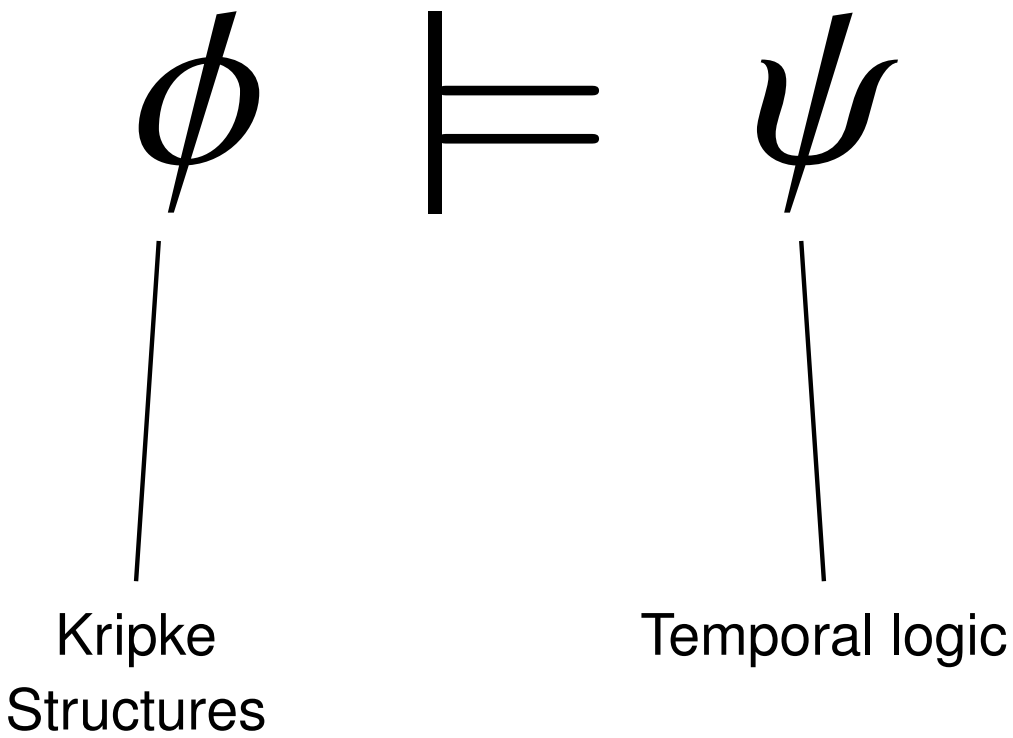
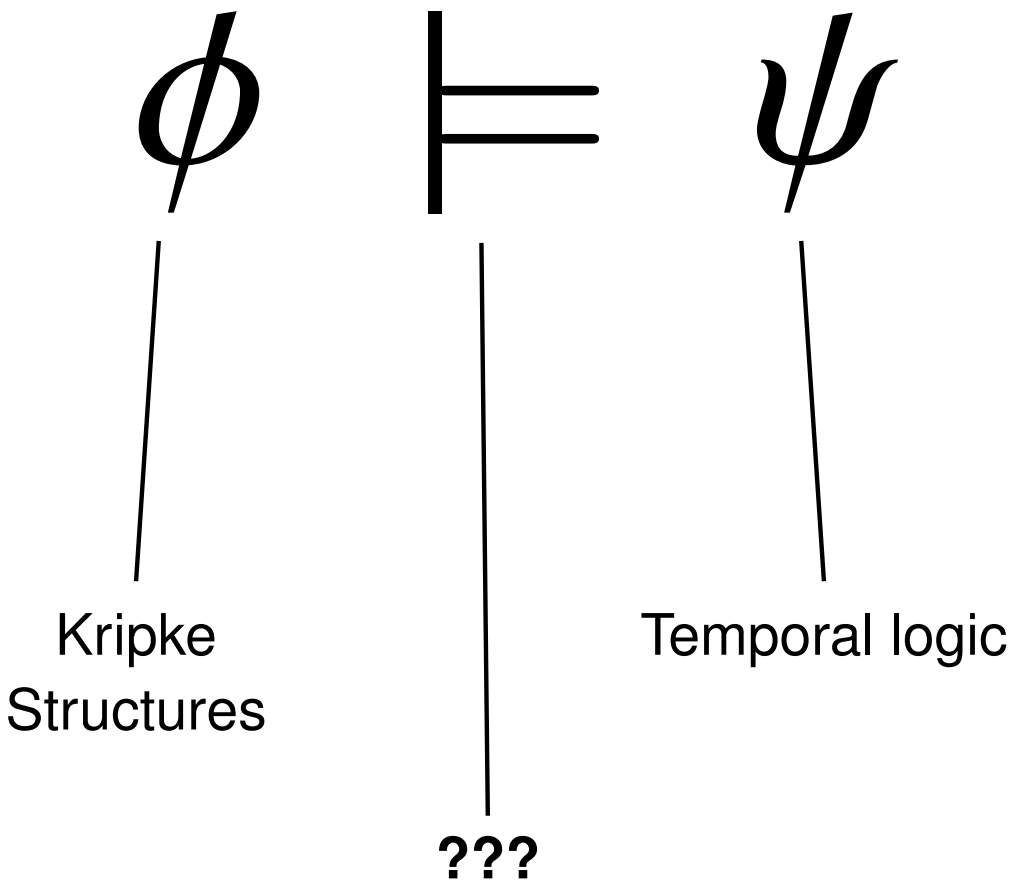
Introduction



Introduction



Introduction

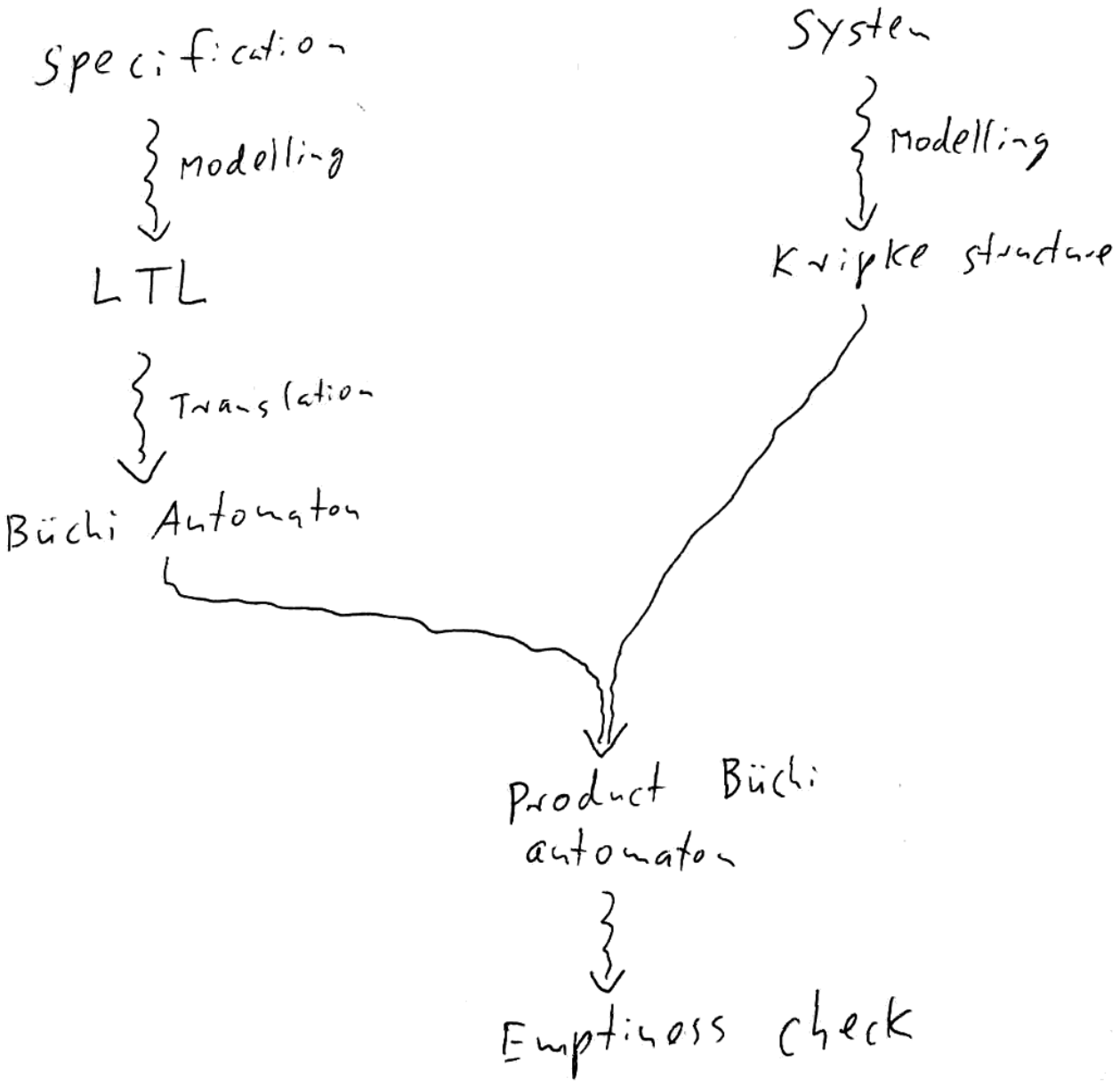


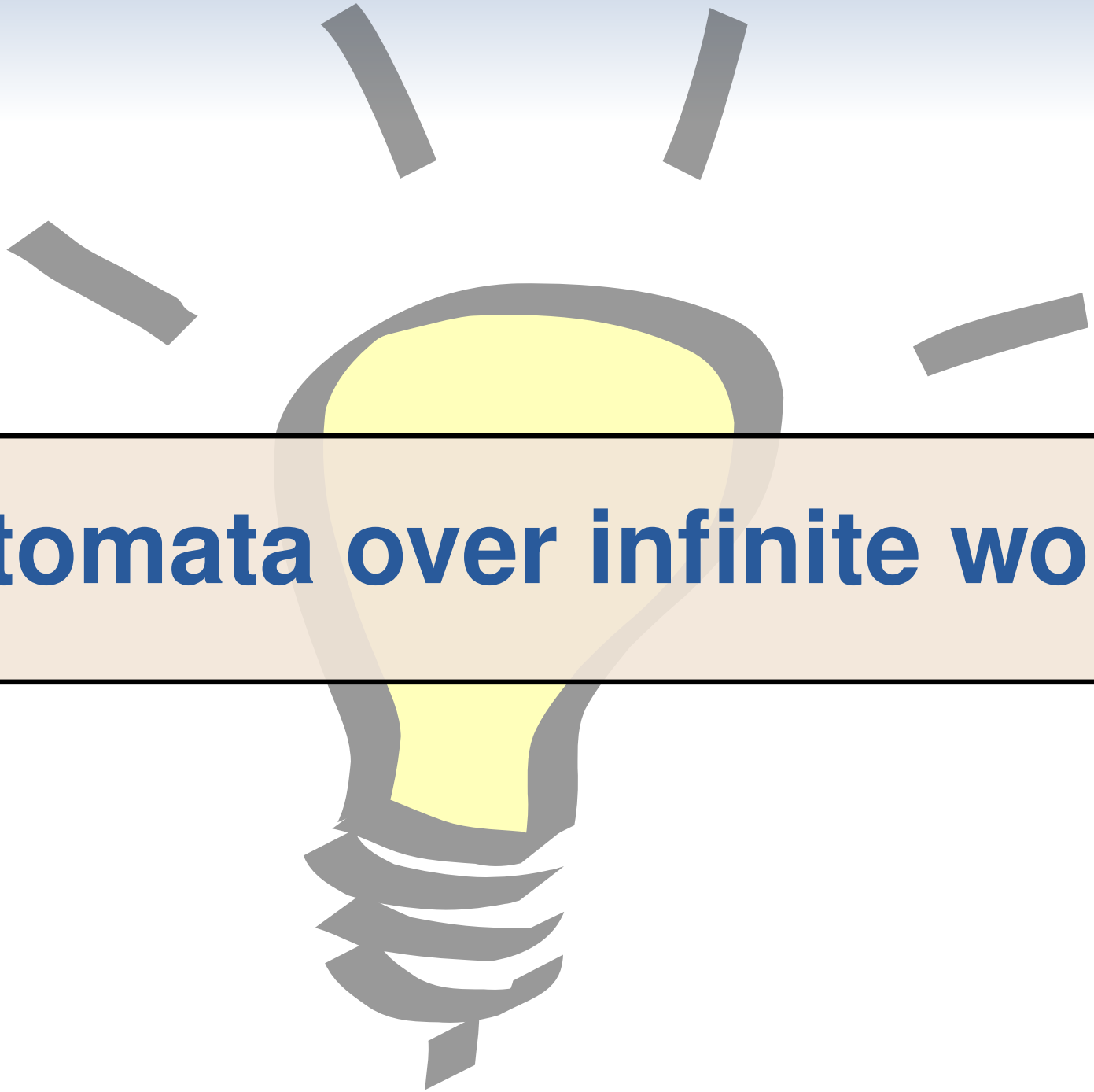
Approach pursued in the following

Steps

1. Translating the specification to an *automaton over infinite words*.
2. Building the product of the automaton and the Kripke structure
3. Checking the product for *language emptiness*

Overview


Specification
modelling
LTL
Translation
Büchi Automaton
System
Modelling
Kripke structure
Product Büchi automaton
Emptiness check



Automata over infinite words

Motivation

Automata over infinite words

- They capture specifications as *languages*
- They are a more “technical” representation for a specification, as needed for model checking.
- Well-researched topic since the 1960’s.

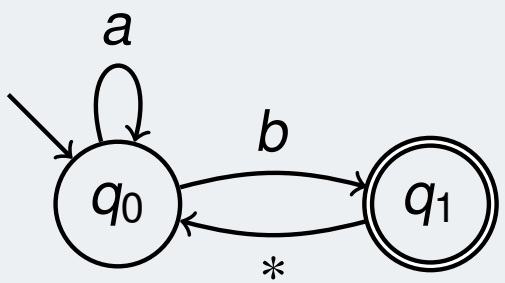
A first automaton type: (Non-deterministic) Büchi automata

Büchi automata

- Introduced by Büchi (1962)
- Have the same syntactic structure as automata over finite words
- *Accept* or *reject* infinite words over some alphabet Σ

Büchi automata by example

Example automaton 1 for $\Sigma = \{a, b\}$



An example *run* of the automaton for a word

$\rho =$												
$\pi =$	$q0$											

Büchi automata by example

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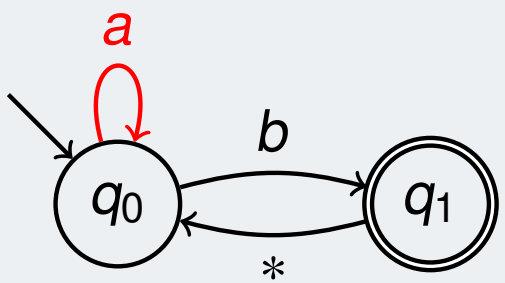

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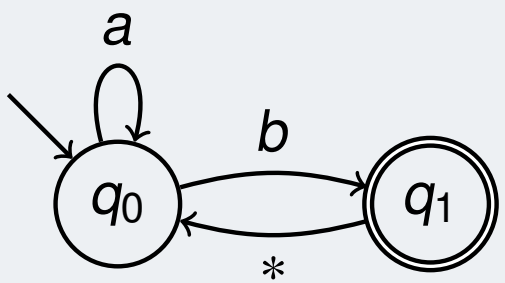

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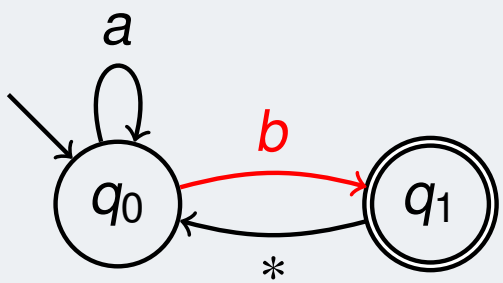

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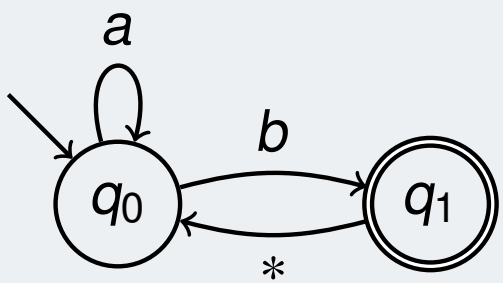

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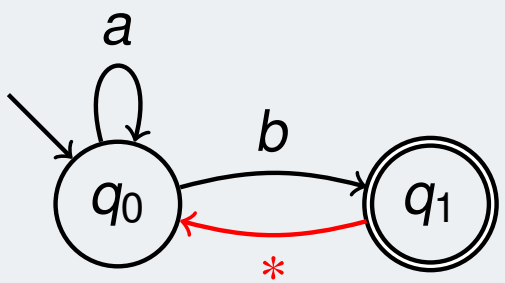

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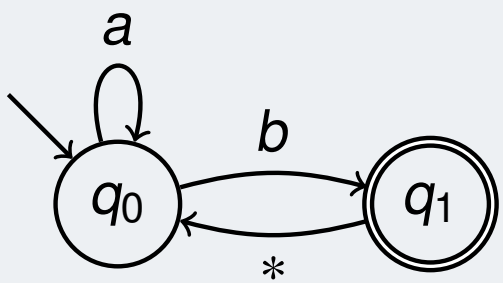

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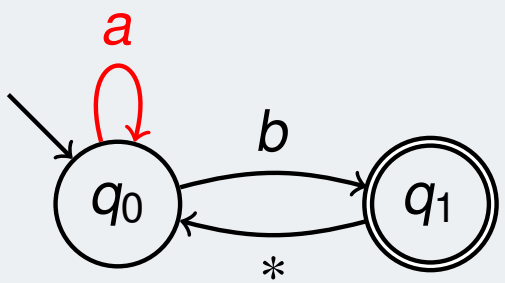

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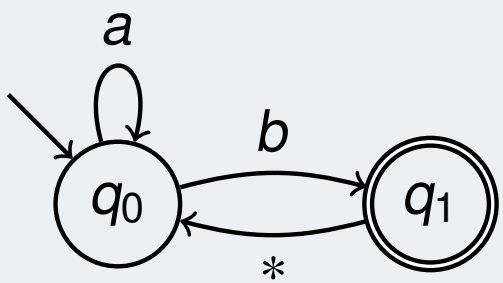

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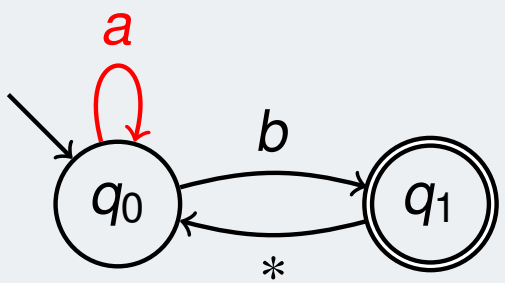

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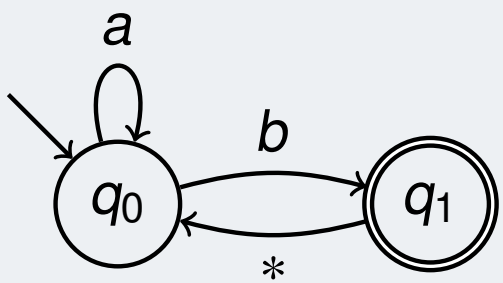

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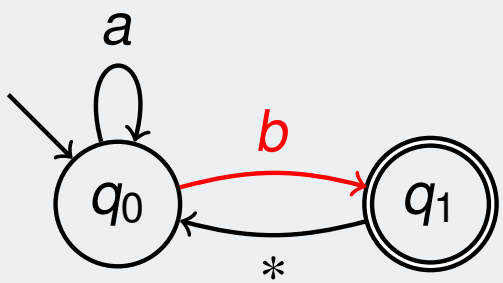

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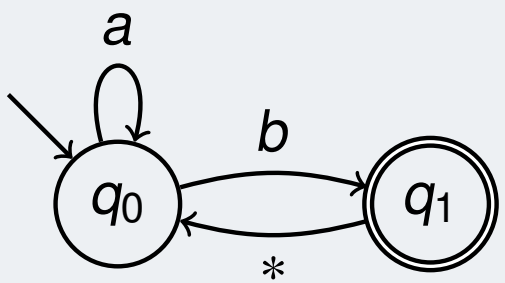

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Büchi automata by example

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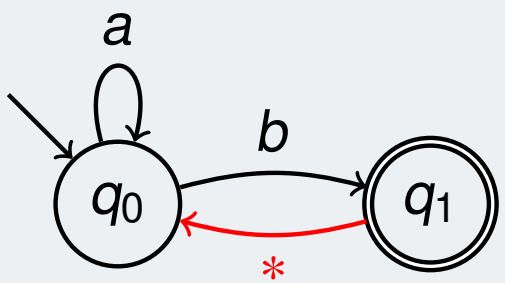

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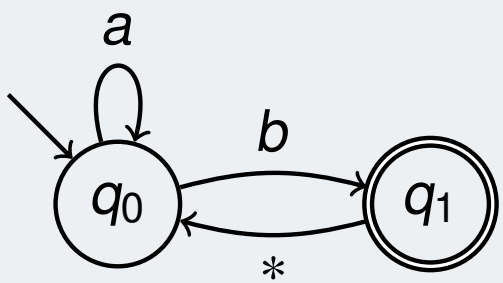

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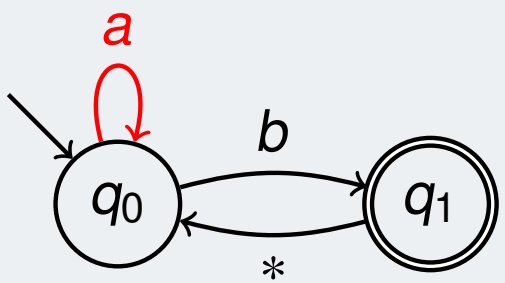

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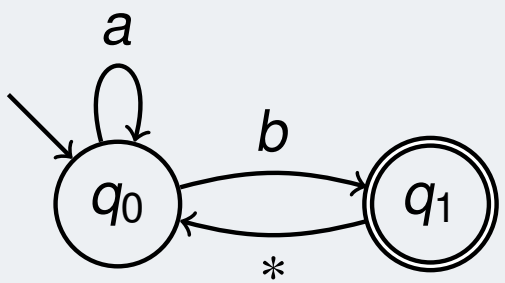

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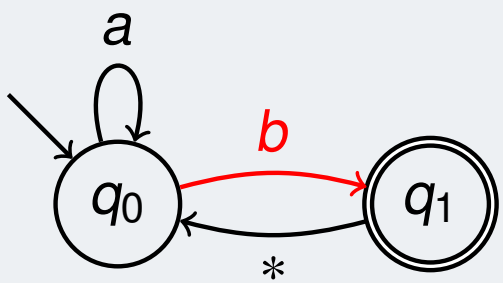

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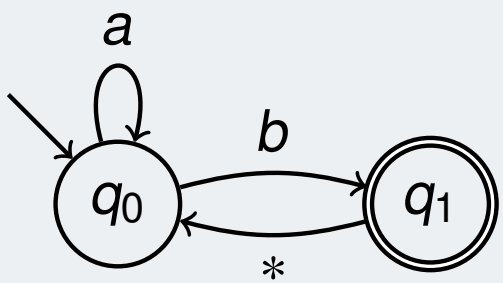

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Büchi automata by example

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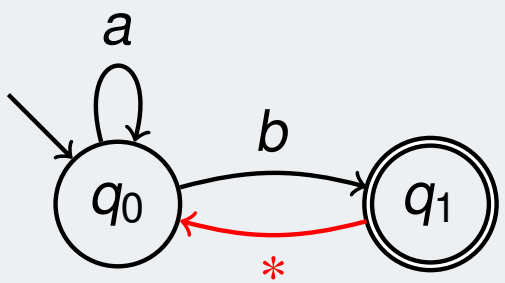

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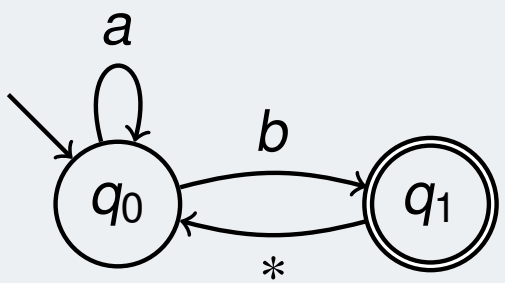

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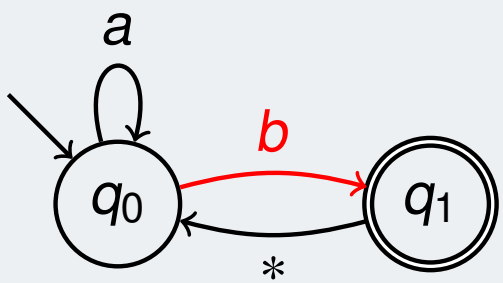

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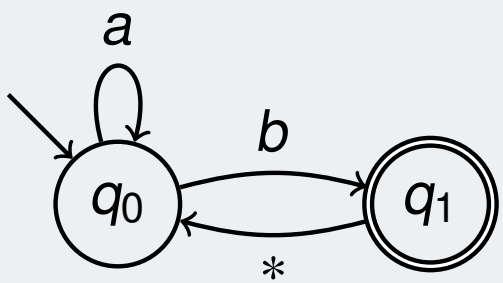

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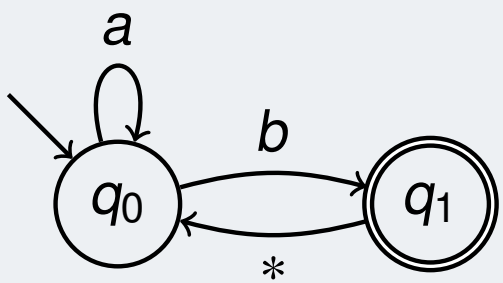

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$\rho =$	a	b	a	a	a	b	a	a	b	b	b	$\ldots$
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Formal definition of Büchi automata (1)

Definition

Given a finite alphabet Σ, a (non-deterministic) Büchi automaton is a tuple $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ with the following components:

- A *finite* set of states Q,
- the alphabet Σ,
- the transition relation $\delta \subseteq Q \times \Sigma \times Q$,
- the initial states $Q_0 \subseteq Q$, and
- the set of *accepting states* $F \subseteq Q$.

Note that these are exactly the same components as for a (non-deterministic) automaton over finite words (NFAs). The similarities between these concepts do not end here, however.

Formal definition of Büchi automata (2)

Definitions

Let $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ be a Büchi automaton and $w = w_0 w_1 \ldots \in \Sigma^\omega$ be an infinite word.

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Likewise, some finite sequence of states $\pi = \pi_0 \pi_1 \ldots \pi_n \in Q^\omega$ is a finite run of $\mathcal{A}$ on w if $\pi_0 \in Q_0$, and for every $0 \leq i < n \in \mathbb{N}$, we have $(\pi_i, w_i, \pi_{i+1}) \in \delta$.

Formal definition of Büchi automata (3)

As for NFAs, runs can be accepting or rejecting, and the automaton accepts all words that have accepting runs. Since we are dealing with words of infinite length here, however, we cannot define the acceptance of such a word over which state is reached last.

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A run $\pi = \pi_0\pi_1 \ldots \in Q^\omega$ of a Büchi automaton $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ is accepting if and only if the set $\{i \in \mathbb{N} \mid \pi_i \in F\}$ is infinitely large.

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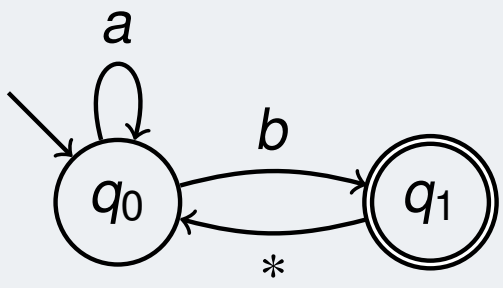
The set of words accepted by the automaton is called its *language*.

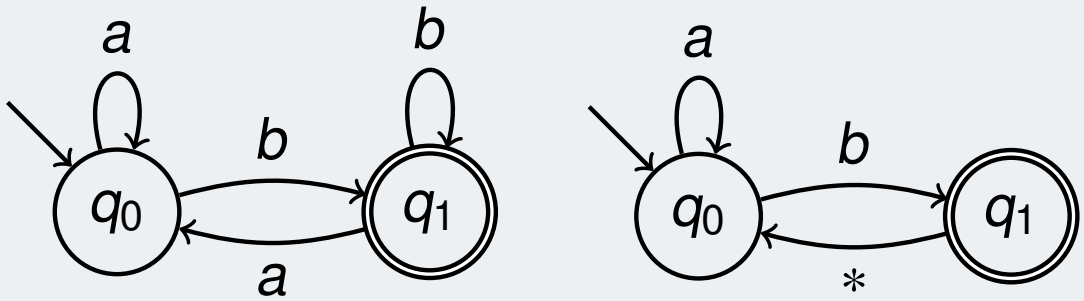
Revisiting the example

Example automata for $\Sigma = \{a, b\}$



Continued example

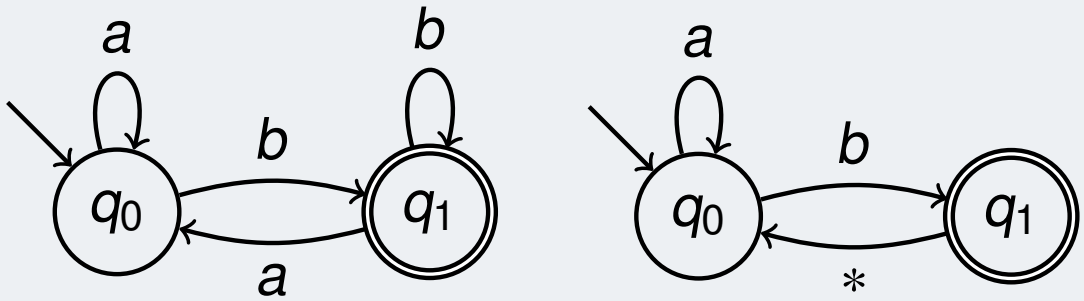
Example automata for $\Sigma = \{a, b\}$



Do they represent different languages?

Continued example

Example automata for $\Sigma = \{a, b\}$

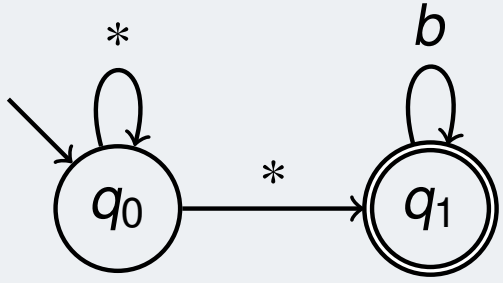


Do they represent different languages?

No, as the automaton still accepts all words in which there are infinitely many *b*s.

Some examples (2)

Example automaton 2 for $\Sigma = \{a, b\}$



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Example 3 for $\Sigma = \{a, b, c\}$

At some point in the future, every sequence of as is immediately followed by a b.

Deterministic vs. non-deterministic Büchi automata (1)

Determinism

We say some that automaton $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ is *deterministic* if for every $q \in Q$ and $x \in \Sigma$, there is at most one state $q' \in Q$ with $(q, x, q') \in \delta$.

Comparison to automata over finite words

While for automata over finite words, deterministic and non-deterministic automata are equally expressive, *deterministic* Büchi automata are less expressive than non-deterministic Büchi automata.

Deterministic vs. non-deterministic Büchi automata (2)

Claim

The language $L = \{wa^\omega \mid w \in \{a, b\}^*\}$ for $\Sigma = \{a, b\}$ is not representable by a deterministic Büchi automaton.

Proof by contradiction (part 1)

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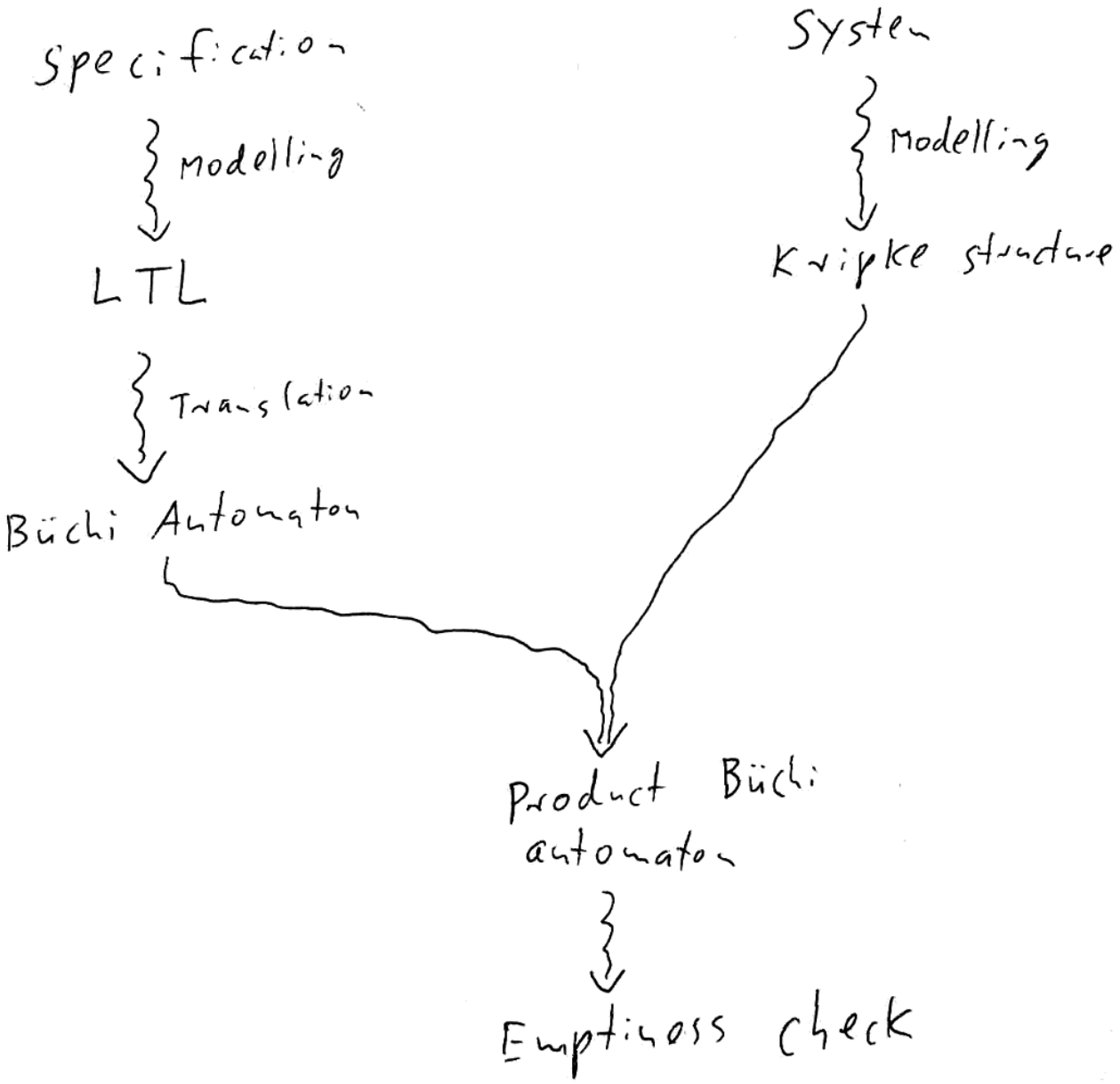
Side note

This also shows that deterministic Büchi automata cannot capture all properties expressible in LTL.

Overview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We will do so in two steps:

- Build a *product* Büchi automaton $\mathcal{B}$ accepting $\text{Traces}(\mathcal{K}) \cap \mathcal{L}(\mathcal{A})$
- Check the product automaton for emptiness.

Product construction – formal definition

Definition

Let $\mathcal{K} = (S, \mathrm{AP}, S_0, T, L)$ be a Kripke structure and $\mathcal{A} = (Q, \Sigma, Q_0, \delta, F)$ be a Büchi automaton with $\Sigma = 2^{\mathrm{AP}}$.

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Proposition (without proof here)

$\mathcal{B}$ accepts exactly those words that accepted by $\mathcal{A}$ **and** are traces of $\mathcal{K}$ at the same time.

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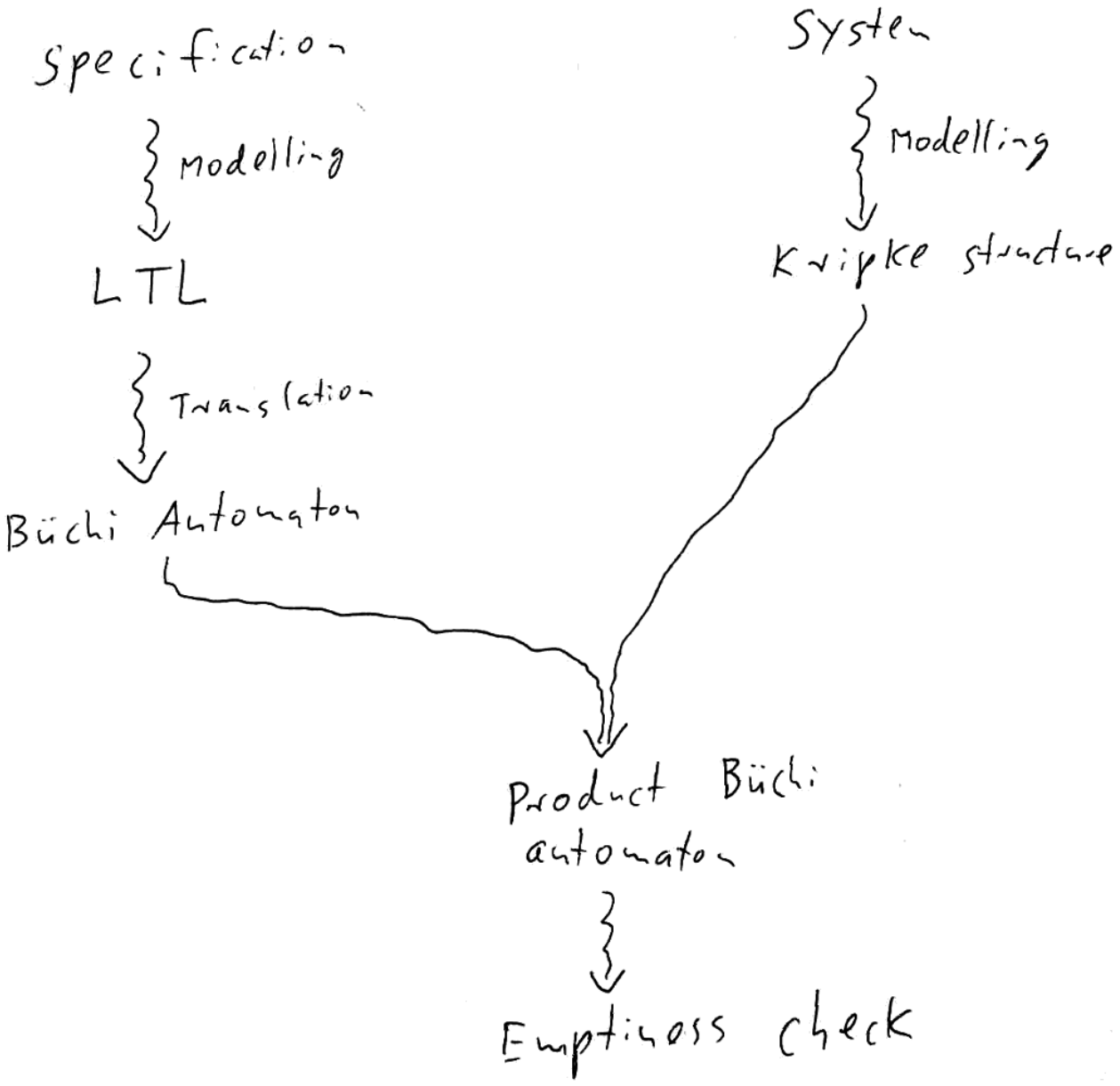
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Checking Büchi automata language emptiness

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Automata over infinite words

We can test this by checking if there exists an accepting *lasso*.

A lasso consists of a *lasso handle* and a *lasso cycle*, where the lasso cycle contains at least one accepting state.

Checking automaton emptiness in practice

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$\rightarrow$ Cubic time overall (which is not good!)

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Basic idea

We combine two depth-first search procedures to find reachable cycles:

1. The first one for finding a path *to* an accepting state
2. The second one for finding a path *back* to the state

The double-DFS algorithm

Shared variables

```
stack = []
stackB = []
visited = set([])
marked = set([])
```

First DFS

```
def dfsA(node):
if node in visited:
  return
stack.push(node)
visited.add(node)
for all (v, v') ∈ E with v==node:
  dfsA(v')
if  v ∈ F:
  dfsB(v')
stack.pop()
```

Second DFS

```
def dfsB(node):
if node in marked:
  return
stackB.push(node)
marked.add(node)
for all (v,v') in E with v==node:
  if v' in stack:
    lasso found
  dfsB(v):
stackB.pop()
```

Calling dfsa

```
for all v ∈ Q0:
  dfsA(v)
```

Why is the double-DFS algorithm correct? (1/2)

Reasoning (1)

The case that if the algorithm returns `lasso found`, a lasso has been found is relatively clear:

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The first DFS Procedure makes sure that only reachable lasso cycles are found.

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Hence, the assumption that the algorithm did not find a lasso cannot hold.

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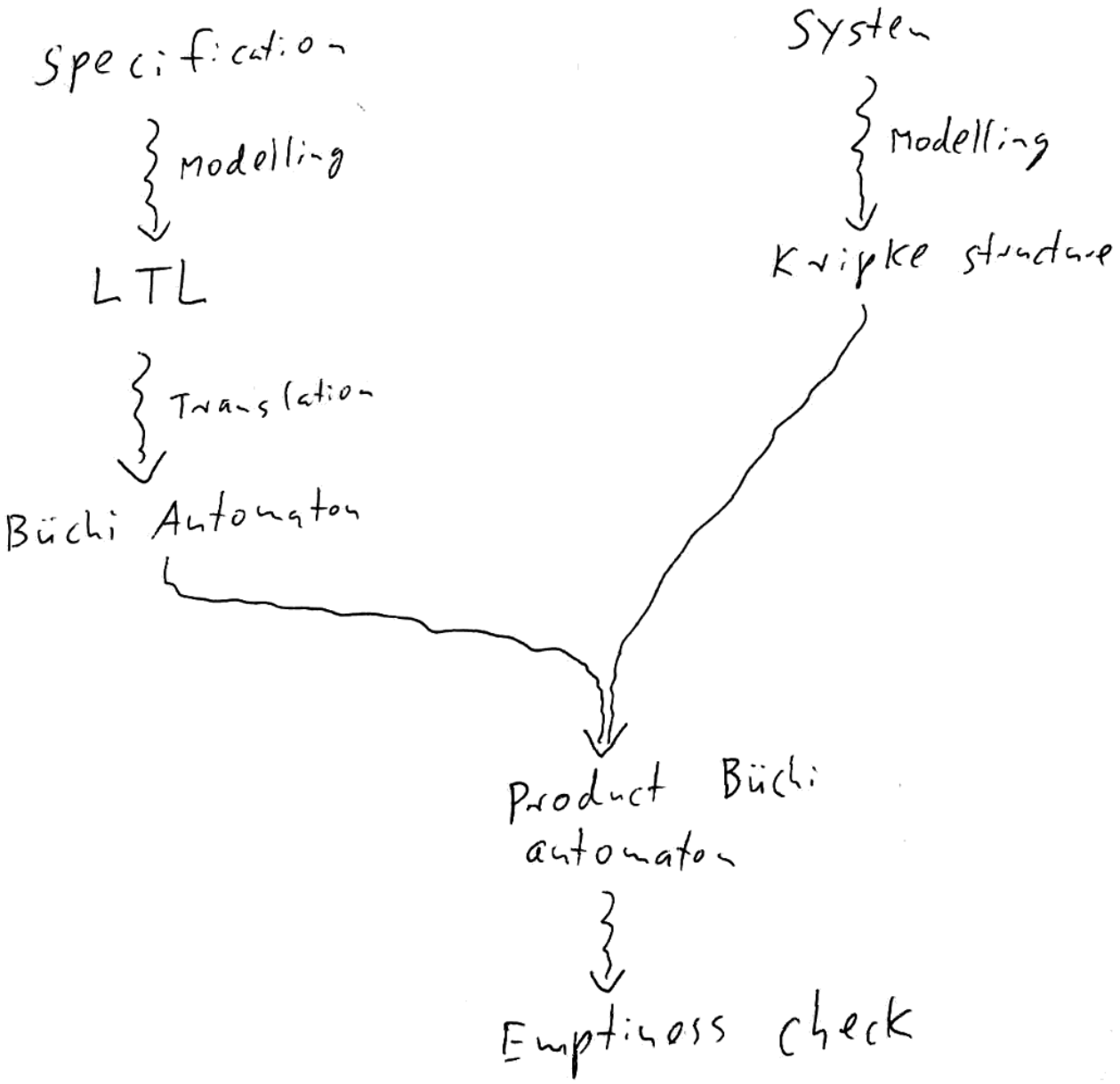
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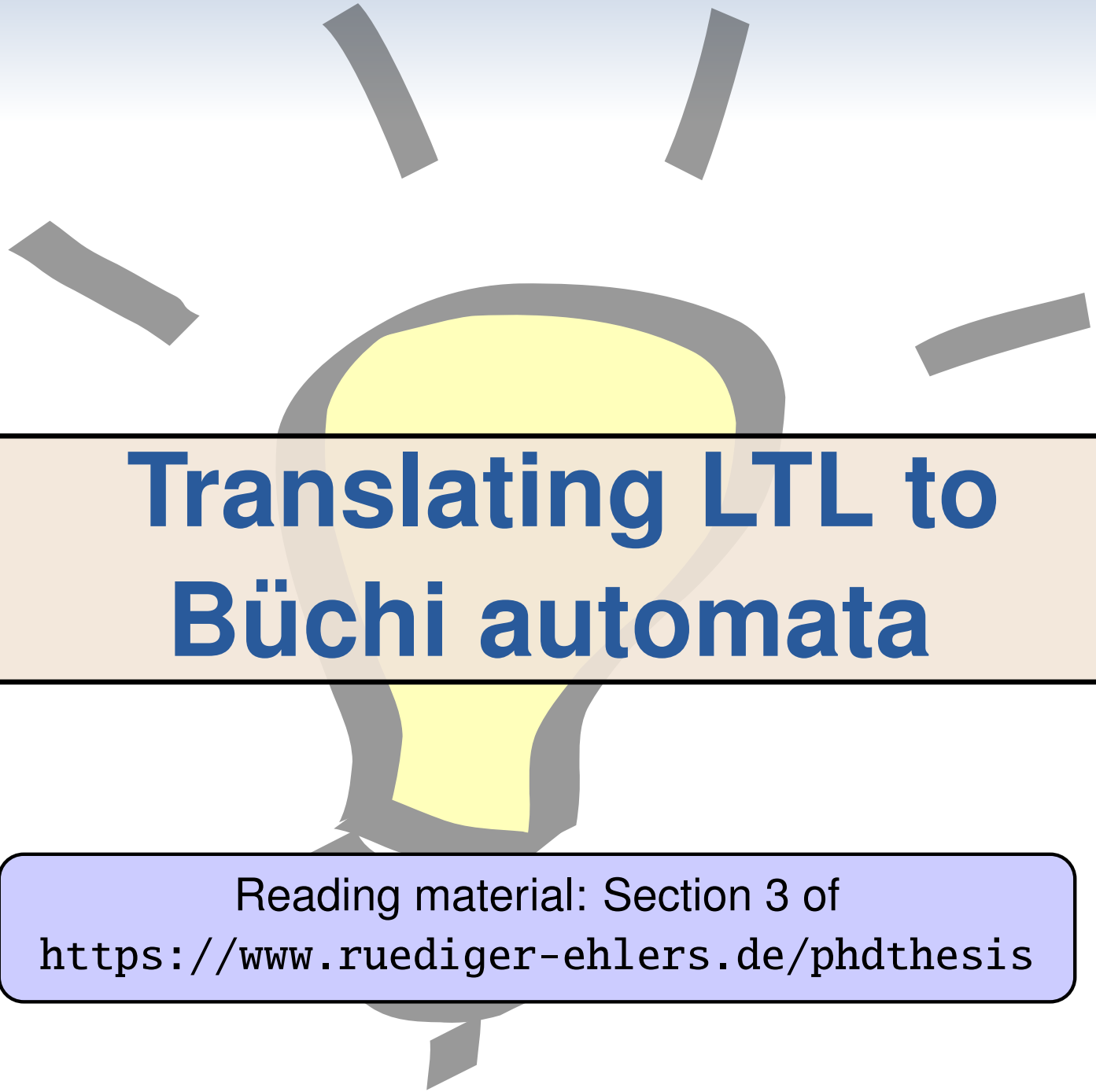

Translating LTL to Büchi automata

Reading material: Section 3 of
https://www.ruediger-ehlers.de/phdthesis

Translating from LTL to Büchi automata

Aim of the translation

Given an LTL formula, we want to translate an LTL formula to a Büchi automaton such that the language of the Büchi automaton is exactly the set of words satisfying the LTL formula.

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Approach presented in the following

We do a multi-step translation:

LTL

$\Downarrow$

LTL in negation normal form

$\Downarrow$

Alternating Büchi automaton

$\Downarrow$

(Non-deterministic) Büchi automaton

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LTL: Translation to NNF

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We utilize some equivalences (for all LTL subformulas ψ):

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We define the *release operator* as the complement of the *until operator*.

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Translation procedure to NNF

$$
\mathit{toNNF}(\psi) = \begin{cases}
\psi & \text{if } \psi \in \{\mathbf{true}, \mathbf{false}\} \\
\psi & \text{if } \psi \in \{\neg p \mid p \in \mathrm{AP}\} \cup \mathrm{AP} \\
\mathit{toNNF}(\psi') \vee \mathit{toNNF}(\psi'') & \text{if } \psi = \psi' \vee \psi'' \\
\mathit{toNNF}(\psi') \wedge \mathit{toNNF}(\psi'') & \text{if } \psi = \psi' \wedge \psi'' \\
\mathbf{G}(\mathit{toNNF}(\psi')) & \text{if } \psi = \mathbf{G}\psi' \\
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$$

Checkpoint!

Steps of the translation

LTL

$\Downarrow$ ✓

LTL in Negation normal form

$\Downarrow$

Alternating Büchi automaton

$\Downarrow$

(Non-deterministic) Büchi automaton

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Idea

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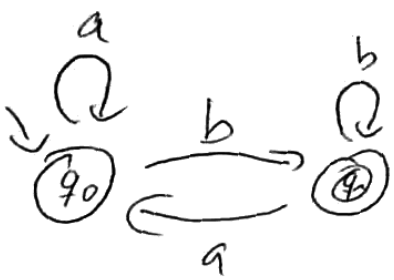
Implementing this idea

We will extend the *branching capabilities* of the automaton.

Branching modes

Det. Aut



One word

a b a b b a ...

One run

$q_0\ q_1 \ldots$

$w \in \mathcal{L}(A)$ if the run accepts

Non-det. Aut.

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Universal aut.

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Representing branching modes in transition functions

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An alternative representation of the transition relation of a non-deterministic Büchi automaton $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ with $\delta \subseteq Q \times \Sigma \times Q$ is that of a Boolean transition function.

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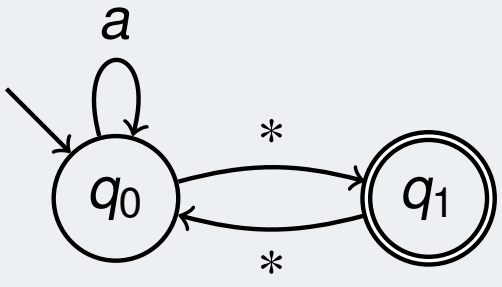

We can either write $\delta = \{(q_0, a, q_0), (q_0, a, q_1), (q_0, b, q_1), (q_1, a, q_0), (q_1, b, q_0)\}$

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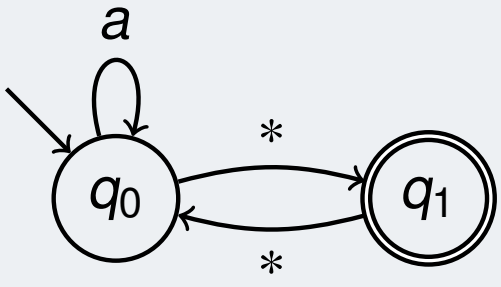

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or

we define $\delta : Q \times \Sigma \to \mathcal{B}^+(Q)$, where $\mathcal{B}^+(Q)$ is the set of positive Boolean formulas over Q and define $\delta(q_0, a) = q_0 \vee q_1$, $\delta(q_0, b) = q_1$, $\delta(q_1, a) = q_0$, $\delta(q_1, b) = q_1$.

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Using positive Boolean formulas, we can represent both *non-deterministic and universal branching*.

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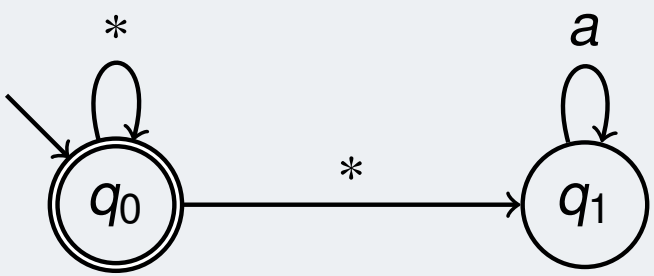
$$\delta(q, a) = q_1 \wedge q_2$$

represents that the run needs for accept *all* possible successors to witness an accepted word.

Example

(Universal) automaton for $\Sigma = \{a, b\}$



Alternative representation of the translation function/relation

$\delta(q_0, a) = q_0 \wedge q_1$
$\delta(q_0, b) = q_0 \wedge q_1$
$\delta(q_1, a) = q_1$ $\delta(q_1, b) =$ **true**

A closer look at universal branching

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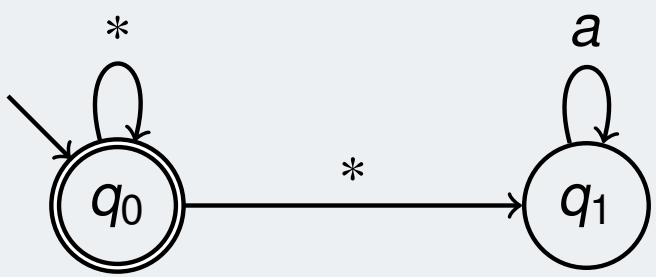
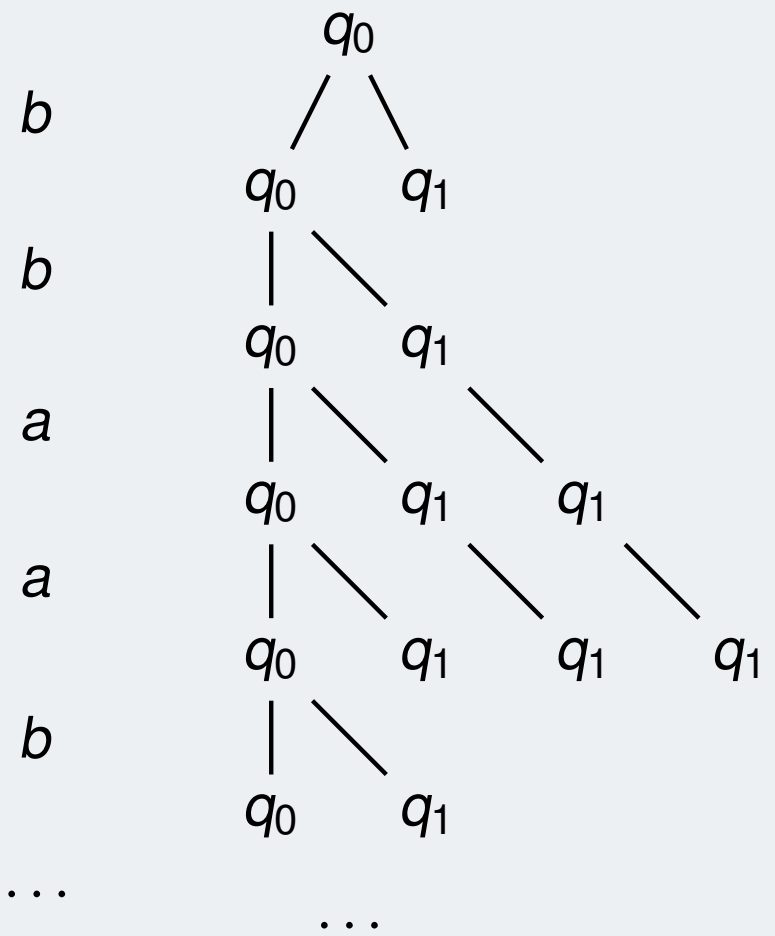


A word and a **run tree**

$w =$



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An alternating Büchi tree automaton accepts all words that have accepting run trees.

A run tree is accepting if all of its infinite branches visit accepting states infinitely often.

Run trees

Given an alternating Büchi automaton $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ and a word $w = w_0 w_1 \ldots \in \Sigma^\omega$, a tree $\langle T, \tau \rangle$ is a **run tree** if it satisfies the following conditions:

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- $\tau(\epsilon) \in Q_0$

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Given an alternating Büchi automaton $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ and a word $w = w_0 w_1 \ldots \in \Sigma^\omega$, a tree $\langle T, \tau \rangle$ is a **run tree** if it satisfies the following conditions:

- $T \subseteq \mathbb{N}^*$, i.e., $\mathbb{N}$ is the set of directions of $\langle T, \tau \rangle$
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Example

When translating the LTL formula $a \vee \mathbf{X}(a \,\mathcal{U}\, b)$ we will have states for the subformulas:

- a
- b
- $a \,\mathcal{U}\, b$
- $\mathbf{X}(a \,\mathcal{U}\, b)$
- $a \vee \mathbf{X}(a \,\mathcal{U}\, b)$

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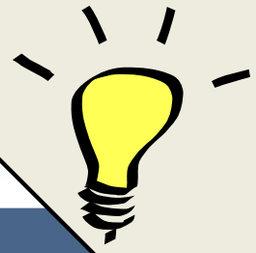

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We define the states for all **G** and **R** subformulas as accepting, the rest is non-accepting. The state for the formula we started with is the only initial state.

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So far

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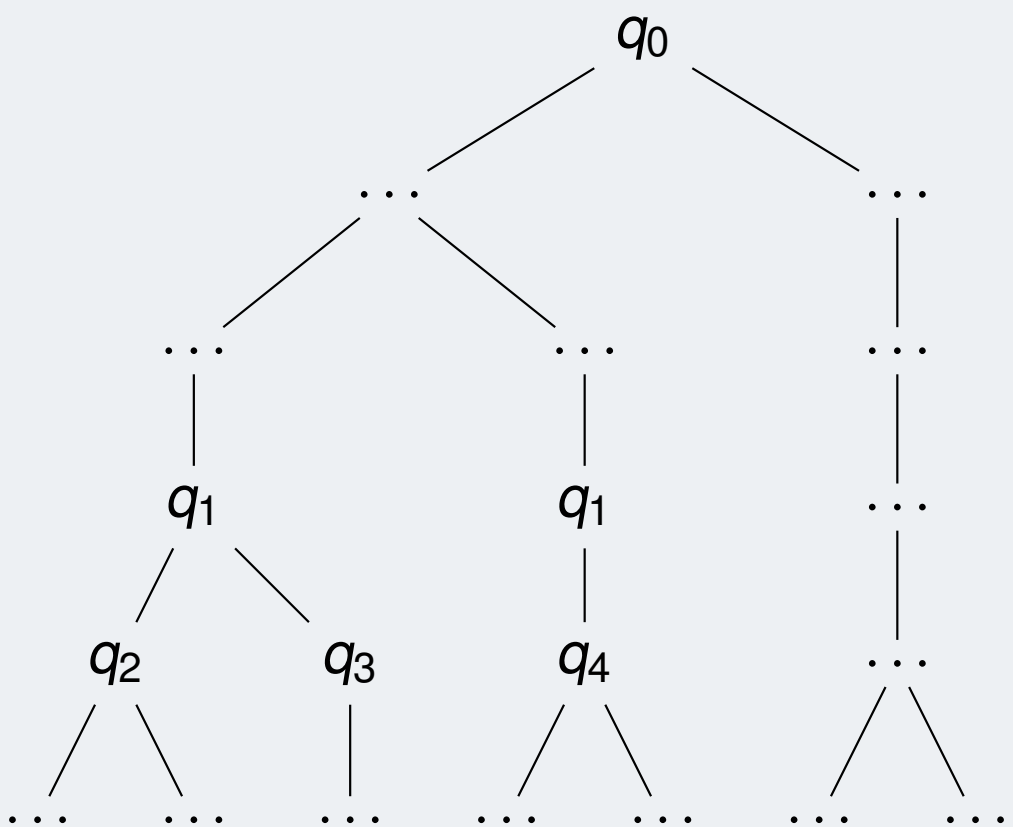
For this to work, the run tree need to have a special shape.

(Non-)Uniform run trees

A non-uniform run tree



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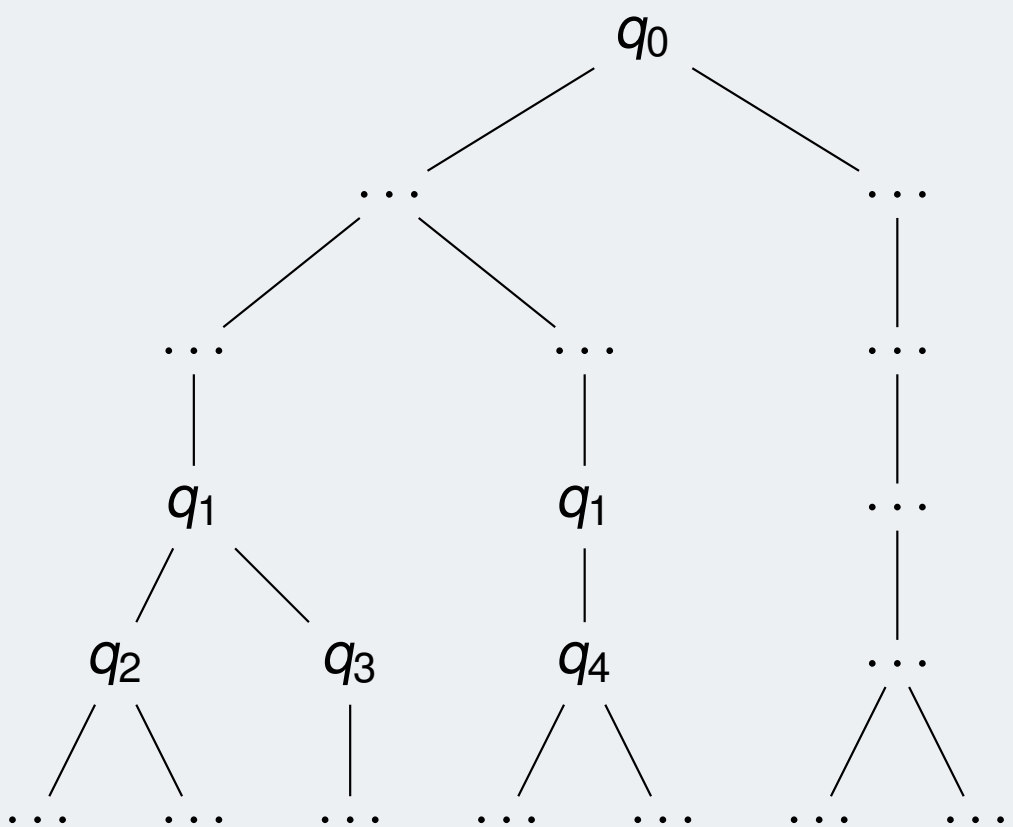

Definition of a uniform run tree

Formally, we call a run tree $\langle T_r, \tau_r \rangle$ *uniform* if for every two nodes t_1 and t_2, we have that if $|t_1| = |t_2|$ and $\tau(t_1) = \tau(t_2)$, then the sub-trees induced by t_1 and t_2 are the same.

Run trees

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We can translate an accepting run tree by iterating through the levels and replacing subtrees by other subtrees. Since all subtrees are accepting, the tree stays accepting.

The Miyano-Hayashi construction

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If and only if we visit break-points infinitely often, the run of the non-deterministic automaton is accepting and all branches in the corresponding run tree of the alternating automaton are accepting.

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Description

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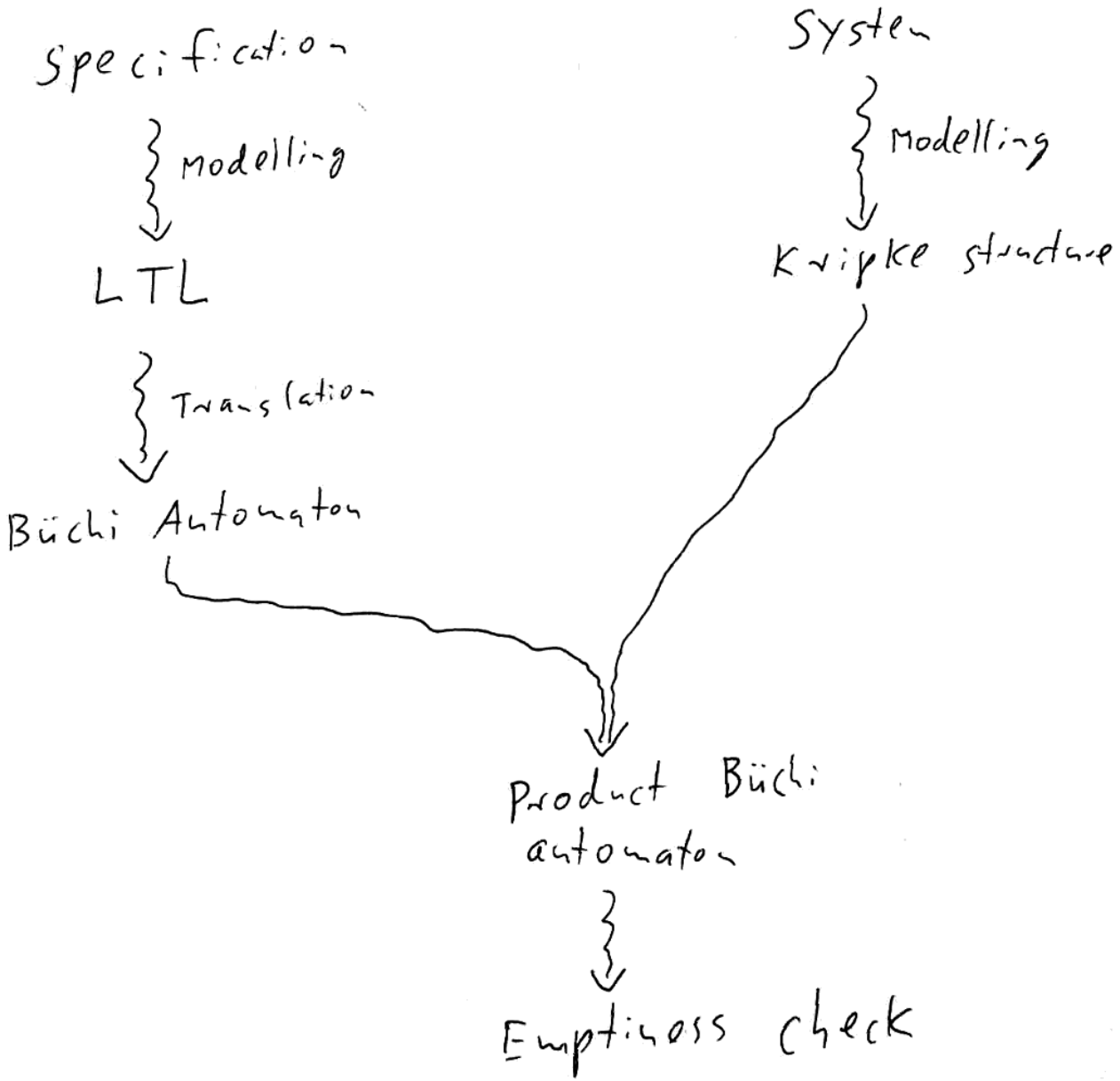
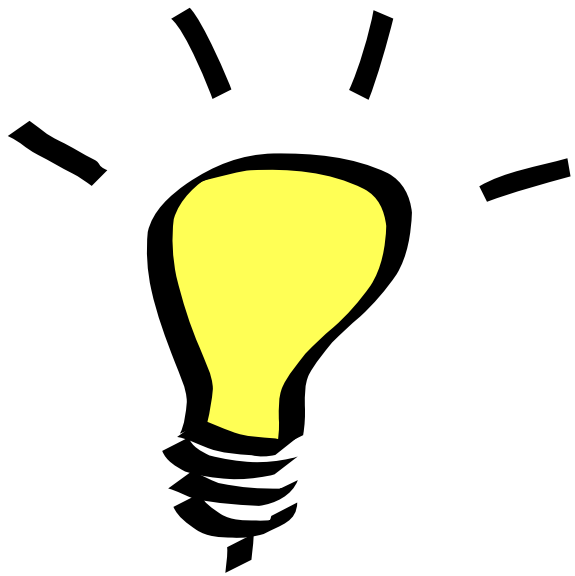
- $Q_0' = \{(\{q_0\}, \emptyset)\}$
- $\mathcal{F} = \{(S, S') \mid S' = \emptyset\}$
- $\delta((S, S'), x) = \{(S'', S''') \mid \forall q \in Q : (q \in S) \rightarrow (S'' \models \delta(q, x)) \wedge (q \in S') \rightarrow ((S''' \cup (S'' \cap \mathcal{F})) \models \delta(q, x)), S'' \supseteq S'''\}$ for all $S, S' \subseteq Q$ with $S' \neq \emptyset$ and $x \in \Sigma$
- $\delta((S, \emptyset), x) = \{(S'', S''') \mid \forall q \in Q, (q \in S) \rightarrow (S'' \models \delta(q, x)), S''' = S'' \setminus \mathcal{F}\}$ for all $S \subseteq Q$ and $x \in \Sigma$

Overview


Specification
modelling
LTL
Translation
Büchi Automaton
System
Modelling
Kripke structure
Product Büchi automaton
Emptiness check

Summary / List of Concepts

- Linear temporal logic
- Non-deterministic Büchi automata
- Product construction for model checking
- (Efficient) emptiness checking of non-deterministic Büchi automata
- Alternating automata
- Miyano-Hayashi construction



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